Estimating Volatility

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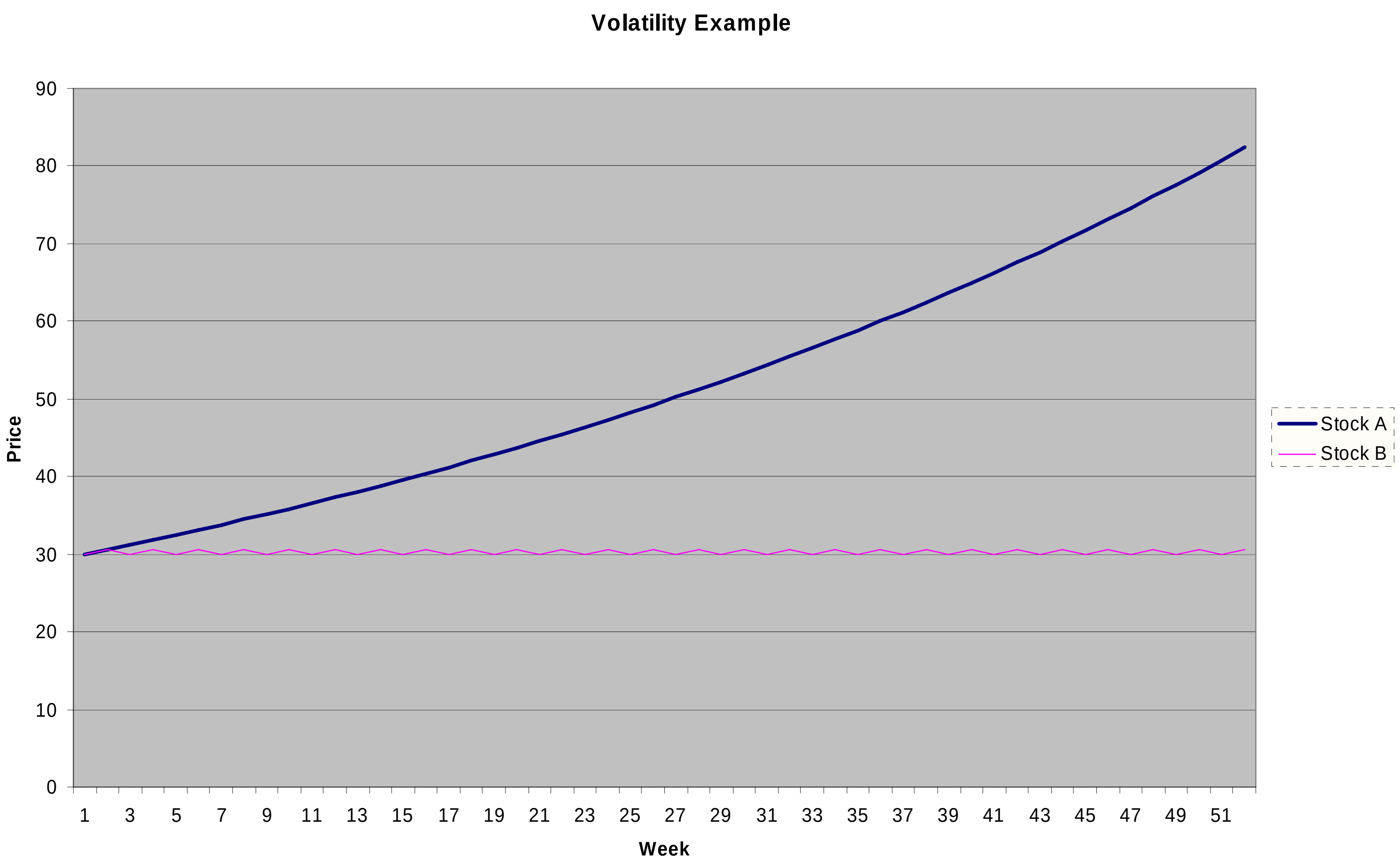
- Definition of Volatility
- Historical Volatility as a Forecast of the Future
- Definition of Implied Volatility
- The Perspective of FAS123R
- The Perspective of SAB 107

Guess the Volatility



Intuition

- Two companies stock price increased from $50 per share to $78.08 per share over three years.
- The average weekly return is $0.18.
- Absolute spread is the average difference between the actual weekly return and $0.18.
- Volatility is similar but instead uses the square root of the sum of squared differences.

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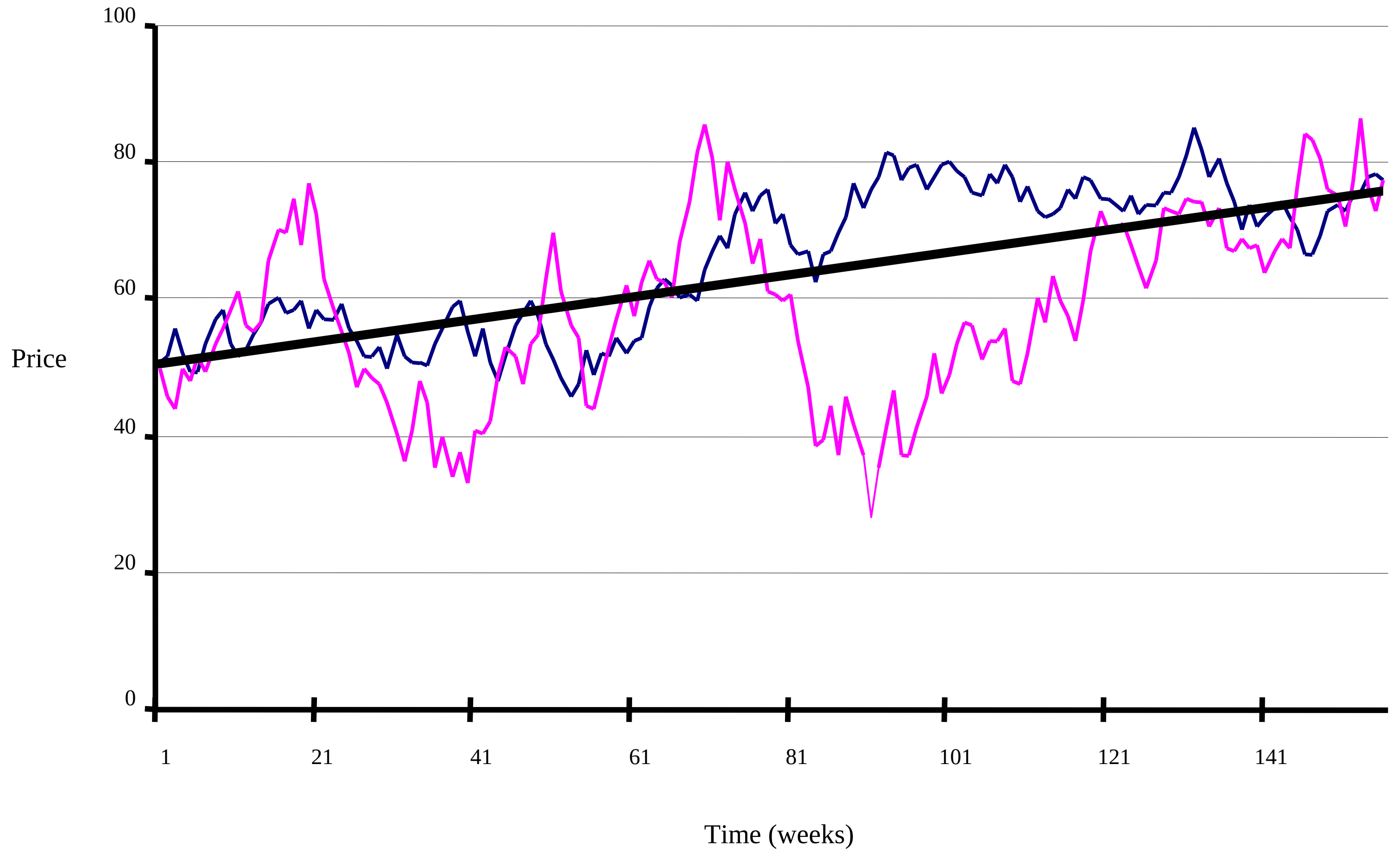
Intuition



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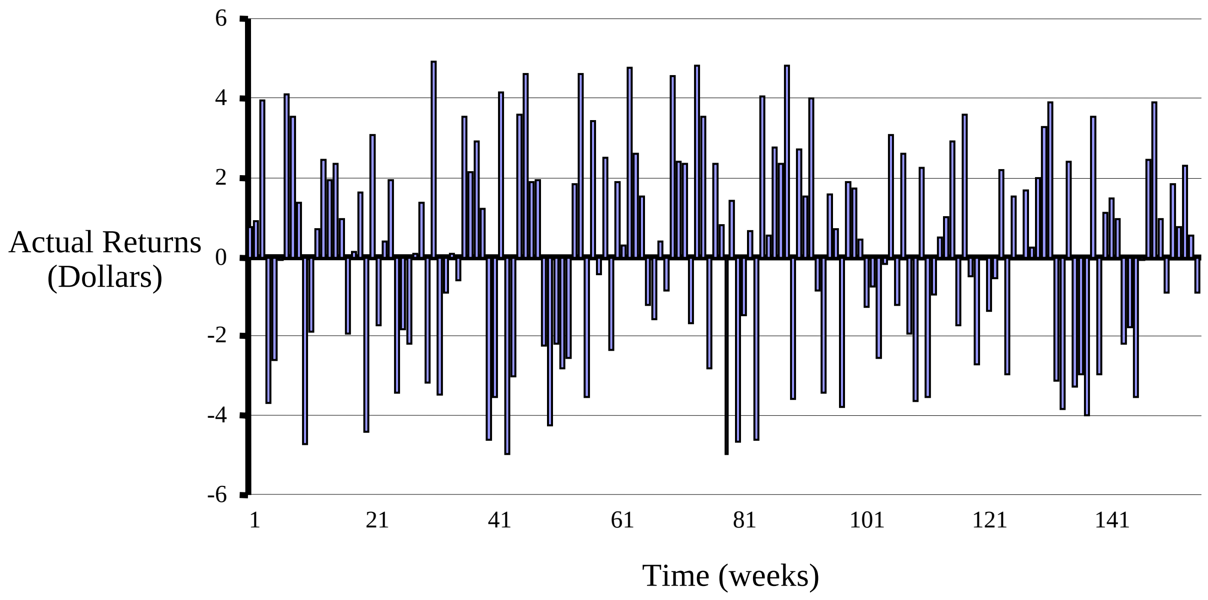
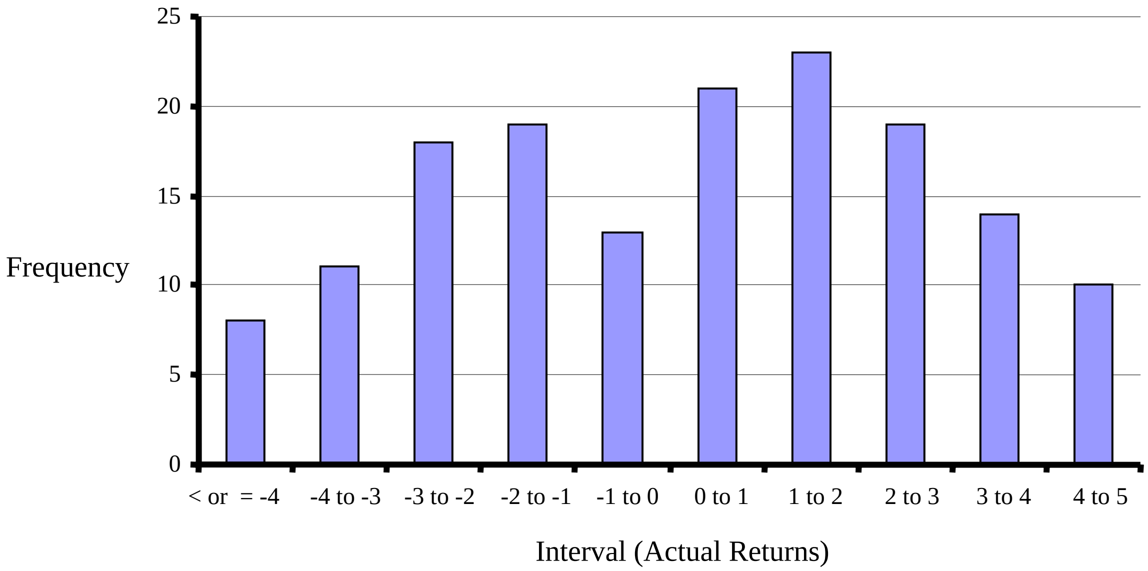
Actual Dollar Returns



Frequency of Dollar Returns



Definition of Volatility

Volatility is the standard deviation of stock returns over a period of time.

It is a measure of the spread of stock returns relative to a central trend or drift.

Computing Historical Volatility

Form the log returns:

$$R_i = \ln(S_i / S_{i-1}) = \ln(S_i) - \ln(S_{i-1})$$

Compute the mean:

$$\overline{R} = \frac{1}{n} \sum_{i=1}^{n} R_i$$

Take the Std. Deviation:

$$\sigma_n = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} \left(R_i - \overline{R}\right)^2}$$

Annualize:

$$\sigma = \sigma_n * \sqrt{\frac{1}{sample\ frequency\ in\ years}}$$

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Intuition

Volatility can be thought of as follows:

It is roughly the percentage range of returns on either side of the market’s expectation that the stock will fall with probably 2/3.

- If the expected return is 10% and the volatility is 40%, then the market thinks the return is 2/3 likely to fall between – 30% and 50%.

FAS 123R on Volatility

From Paragraph A31:

"Volatility is a measure of the amount by which a financial variable, such as share price, has fluctuated (historical volatility) or is expected to fluctuate (expected volatility) during a period. Option-pricing models require an estimate of expected volatility as an assumption because an option's value is dependent on potential share returns over the option's term. The higher the volatility, the more the returns on the share can be expected to vary – up or down."

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What is Expected Volatility?

Finance isn't Physics
from Emanuel Derman's Guide to the Perplexed:

"Physics aspires to fundamental models, with predictions of eight decimal places… Financial modeling is *never* going to provide eight decimal place forecasting... In physics, you're playing against G-d, and G-d doesn't change the laws very often… In finance, you're playing against G-d's creatures, agents who value assets based on their feelings about the future in general and their future in particular; these feelings are ephemeral, or at best unstable, and fresh news on which they are based keeps streaming in. Finance uses parameters like future risk and future return, which not only refer to the future rather than today, but also are opinions rather than facts….

Expected value in finance clearly derives from human beings who are doing the expecting, while mass and charge and electromagnetic force apparently don't - or at least not in an obvious way."

What is Expected Volatility?

- *Expected volatility* is an estimate or forecast of future volatility from the option grant date forward over the term of the option.

- Valuation with Black-Scholes requires a constant expected volatility corresponding to the expected term.

- Valuation with a lattice or simulation model requires an expected volatility term structure corresponding to the full contract term.

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Forecasting Volatility

Expected (Forecasted) Volatility is typically based on:

- **Historical Volatility** – as measured by the sample standard deviation of historical returns.
- **Time Series Models** - based on historical returns but using regression analysis which might show how to weight past observations as well as trends such as mean reversion (*e.g.* GARCH models).
- **Implied Volatility -** referring to the volatility that is implied by an option valuation model when it is set equal to the prices of traded options observed in the market.

FAS 123R on Expected Volatility

Paragraph B86 states that the objective in estimating expected volatility is to ascertain the assumption about expected volatility that marketplace participants would likely use in determining an exchange price for an option.

This suggests that implied volatility would suffice if options with terms less than and equal to employee stock option terms were very actively traded.

Implied Volatility

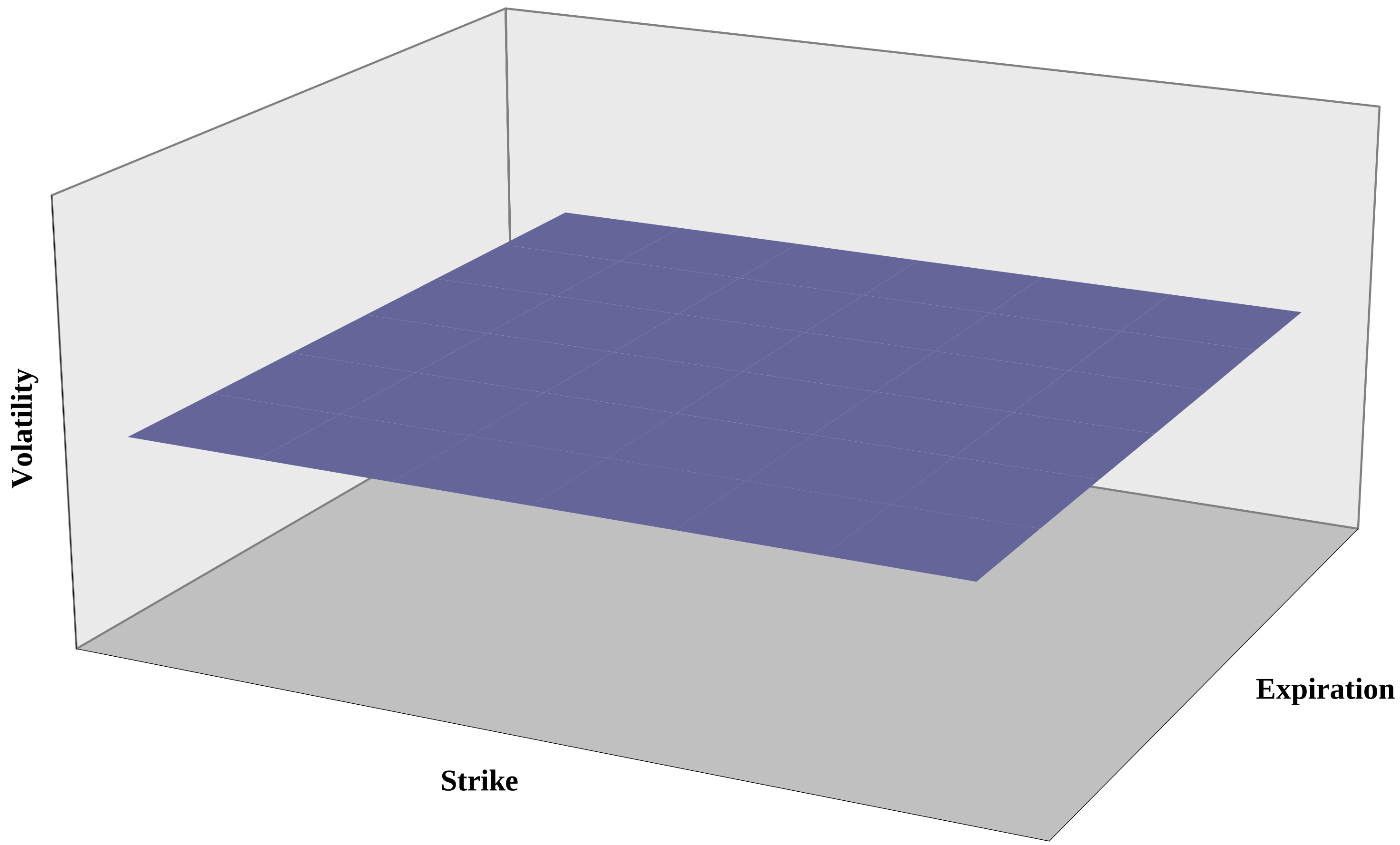
Definition: Implied volatility is inferred from observed exchange-traded option prices by answering the question: What level of volatility, when input into an option valuation model – typically Black-Scholes – would generate the observed market prices?

This process of reverse engineering volatility from prices is possible for two reasons:

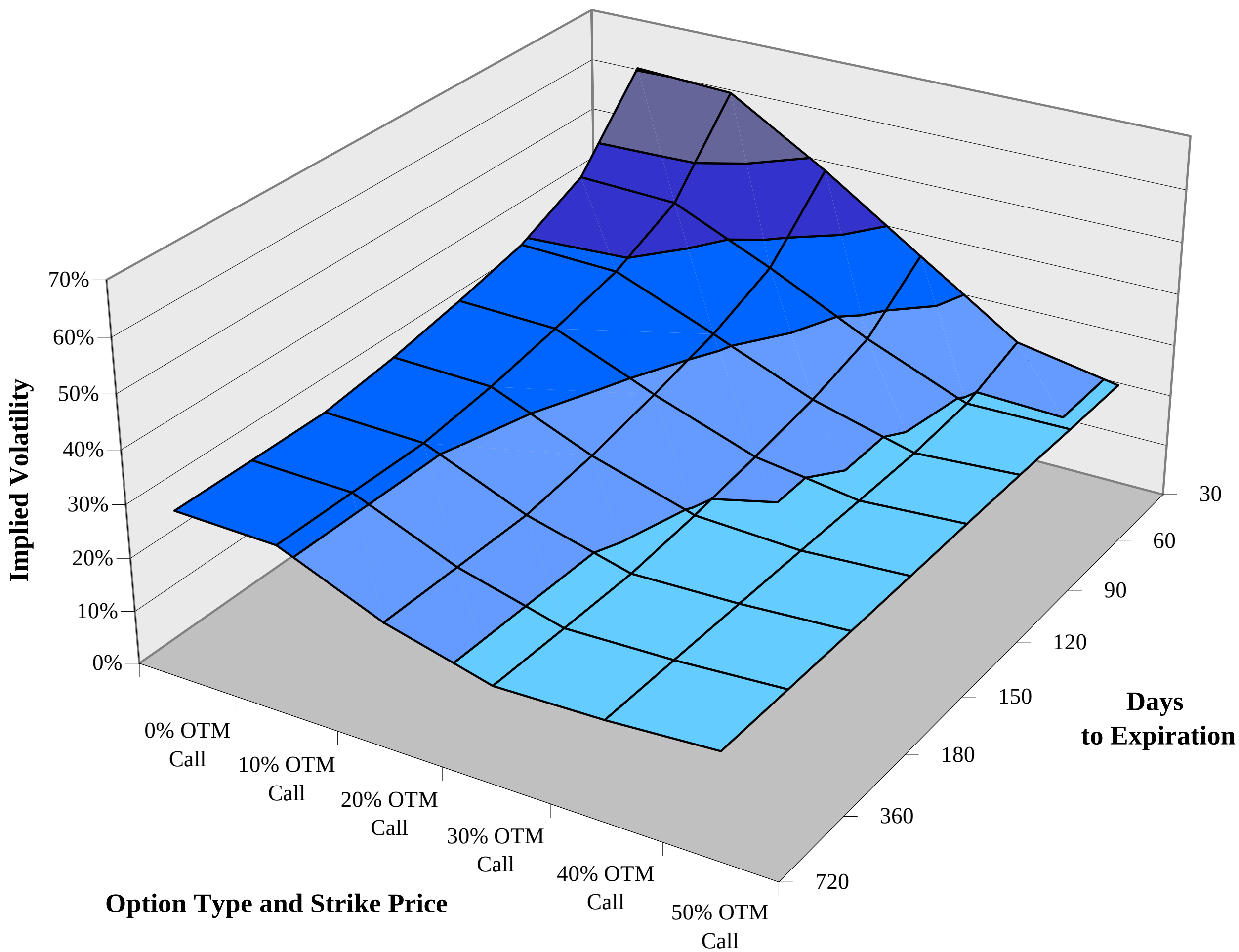
(1) the other parameters that are input into the model (stock and exercise price, option term, risk-free rate and dividend yield) are observable

(2) given the other inputs, option market prices are strictly increasing in the volatility parameter.

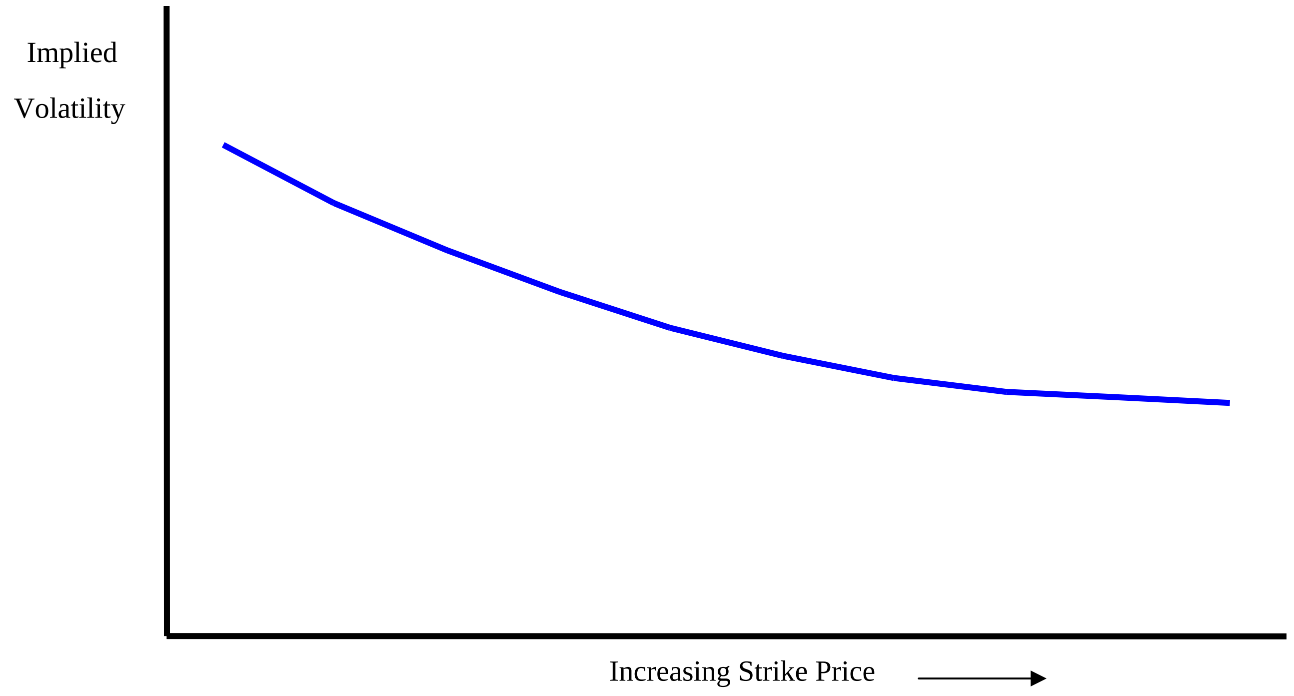
Black-Scholes Volatility



Implied Volatility Surface



The Option Smile is a Skew for Equity Options



FAS 123R on Historical Volatility as a Proxy for Expected Volatility

Paragraph A21 states that companies should begin with historical volatility, but then consider how expected volatility differs from history:

From Paragraph A21:

"Historical experience is generally the starting point for developing expectations about the future. Expectations based on historical experience should be modified to reflect ways in which currently available information indicates that the future is reasonably expected to differ from the past."

FASB on History as a Proxy for Expected Volatility in its March 2004 exposure draft

"An entity that uses historical share price volatility as its estimate of expected volatility without considering the extent to which future experience is reasonably expected to differ from historical experience (and the other factors cited in this paragraph) would not comply with the requirements of this Statement."

FASB on Expected Volatility in a Lattice

"In a lattice model, the assumptions used are to be determined for a particular node (or multiple nodes during a particular time period) of the lattice and not over multiple periods"

FASB on Factors to Consider in Estimating Volatility

From Paragraph A32:

Factors to consider in estimating expected volatility include:

a. Historical volatility, including changes in that volatility and possible mean reversion of that volatility, over the most recent period that is generally commensurate with the contractual term of the option if a lattice model is being used to estimate fair value or the expected term of the option if a closed-form model is being used.

b. Implied volatility

c. The length of time an entity has been public. If less than the expected term the term structure of volatility for the longest period for which trading activity is available should be more relevant. A newly public entity also might consider the expected volatility of similar entities. A nonpublic entity might base its expected volatility on the expected volatilities of entities that are similar except for having publicly traded securities.

d. Appropriate and regular intervals for price observations. A publicly traded entity would likely use daily price observations, while a nonpublic entity might use monthly price observations.

e. Corporate and capital structure (*e.g*. highly leveraged entities tend to have higher volatilities).

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FAS 123R on Nonpublic Companies

A nonpublic entity:

"…if it is not possible to reasonably estimate the fair value of equity share options and similar instruments because it is not practicable to estimate the expected volatility of the entity's share price, a nonpublic entity is required to measure its awards of equity share options and similar instruments based on a value calculated using the historical volatility of an appropriate industry sector index instead of the expected volatility of its share price."

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SEC – SAB 107

- “The staff believes that companies should make good faith efforts to identify and use sufficient information in determining whether taking historical volatility, implied volatility or a combination of both into account will result in the best estimate of expected volatility.”

- “The staff believes that a company with actively traded options or other financial instruments with embedded options generally could place greater (or even exclusive) reliance on implied volatility.”

SEC – SAB 107 Implied Volatility

- Matching option term is desirable, but at minimum company needs some options with terms greater than six months
- If all that is available are options with less than one year maturity, then company should take other information into consideration
- Greater than one year maturity is not expected to be significantly different than less than one year (true?)
- Match volatility to estimate to the date of grant if possible

SAB 107 – Sufficiency of Exclusive Use of Historical Volatility

- If there is no reason to believe future volatility will differ
- Historical volatility is measured as a simple average
- If the period is at least equal to expected term or contractual life
- If there are a reasonably sufficient number of measurements at consistent points

The Large Impact of Volatility on Value

Comparative Valuations - Sensitivity to Volatility

Market Inputs:	Stock	$50.00		Risk Free Rate	4%			Dividends	0%	
Contract Inputs:	Strike	$50.00		Maturity	10			Vesting Period	1	

Binomial Option Value Based on Expiration Rate (and Term Implied by the Valuation)

	Low Probability of Early Exercise		Medium Low Probability of Early Exercise		Medium Probability of Early Exercise		Medium High Probability of Early Exercise		High Probability of Early Exercise	
Departure Rate:	5%	10%	5%	10%	5%	10%	5%	10%	5%	10%
Volatility										
25%	$13.99	$13.22	$12.80	$12.20	$12.20	$11.67	$11.31	$10.88	$10.56	$10.21
	4.78	4.35	4.28	3.92	4.04	3.71	3.70	3.41	3.42	3.17
30%	$15.57	$14.77	$14.30	$13.66	$13.65	$13.09	$12.70	$12.25	$11.90	$11.53
	4.77	4.33	4.29	3.92	4.06	3.72	3.73	3.44	3.47	3.20
35%	$17.15	$16.30	$15.78	$15.11	$15.08	$14.49	$14.07	$13.60	$13.23	$12.84
	4.77	4.33	4.31	3.93	4.08	3.74	3.77	3.46	3.52	3.24
40%	$18.68	$17.80	$17.22	$16.53	$16.47	$15.86	$15.41	$14.92	$14.51	$14.11
	4.77	4.33	4.33	3.95	4.11	3.76	3.81	3.50	3.57	3.28
45%	$20.18	$19.26	$18.63	$17.91	$17.84	$17.21	$16.72	$16.21	$15.78	$15.36
	4.78	4.34	4.35	3.97	4.14	3.78	3.85	3.53	3.62	3.33
50%	$21.63	$20.68	$20.00	$19.26	$19.16	$18.51	$17.99	$17.47	$17.01	$16.57
	4.80	4.35	4.38	3.99	4.17	3.81	3.90	3.57	3.67	3.37
55%	$23.03	$22.06	$21.32	$20.57	$20.44	$19.78	$19.23	$18.69	$18.21	$17.76
	4.82	4.37	4.41	4.02	4.21	3.84	3.94	3.60	3.73	3.41
60%	$24.38	$23.39	$22.60	$21.83	$21.68	$21.01	$20.43	$19.88	$19.37	$18.92
	4.84	4.39	4.45	4.05	4.25	3.87	3.99	3.65	3.78	3.46

Black-Scholes Option Value at Implied Expected Term from Above (and Percentage Value Difference)

Volatility										
25%	$14.76	$13.95	$13.83	$13.13	$13.37	$12.71	$12.69	$12.10	$12.12	$11.58
	5.53%	5.56%	8.06%	7.64%	9.61%	8.95%	12.20%	11.13%	14.77%	13.35%
30%	$16.54	$15.68	$15.59	$14.83	$15.11	$14.39	$14.42	$13.76	$13.84	$13.23
	6.18%	6.17%	9.01%	8.50%	10.71%	9.94%	13.51%	12.33%	16.27%	14.70%
35%	$18.33	$17.41	$17.36	$16.53	$16.86	$16.08	$16.16	$15.44	$15.57	$14.89
	6.88%	6.82%	9.99%	9.39%	11.83%	10.96%	14.83%	13.52%	17.74%	16.04%
40%	$20.11	$19.13	$19.12	$18.24	$18.61	$17.77	$17.90	$17.12	$17.30	$16.56
	7.63%	7.51%	11.01%	10.32%	12.99%	12.02%	16.16%	14.73%	19.21%	17.38%
45%	$21.88	$20.85	$20.87	$19.93	$20.36	$19.45	$19.64	$18.79	$19.03	$18.23
	8.40%	8.22%	12.03%	11.25%	14.14%	13.07%	17.46%	15.93%	20.63%	18.69%
50%	$23.62	$22.54	$22.61	$21.61	$22.09	$21.12	$21.36	$20.45	$20.75	$19.88
	9.20%	8.95%	13.05%	12.20%	15.28%	14.12%	18.75%	17.11%	22.02%	19.97%
55%	$25.33	$24.20	$24.32	$23.27	$23.79	$22.78	$23.07	$22.10	$22.45	$21.53
	9.97%	9.68%	14.05%	13.12%	16.38%	15.14%	19.97%	18.25%	23.34%	21.19%
60%	$27.00	$25.83	$25.99	$24.89	$25.47	$24.40	$24.75	$23.73	$24.13	$23.15
	10.75%	10.40%	15.02%	14.02%	17.45%	16.13%	21.15%	19.35%	24.59%	22.37%

The Large Impact of Volatility on Value

- 52% Historical Vol over 5 years
- 50% Historical Vol over 4 years
- 45% Historical Vol over 3 years
- 30% Implied Vol on 2 year Leap
- 35% Using mixture of Spot and Lagged Implied Vols

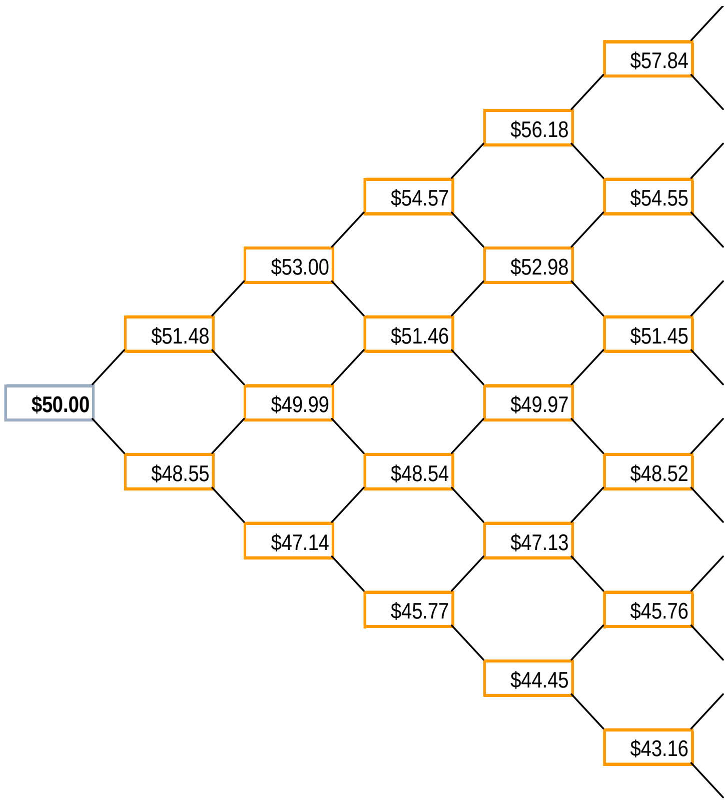
$15.08 at 35% rather than $19.16 at 50% on 10 million shares →

More than $40 million in EBITDA with 35% rather than 50%

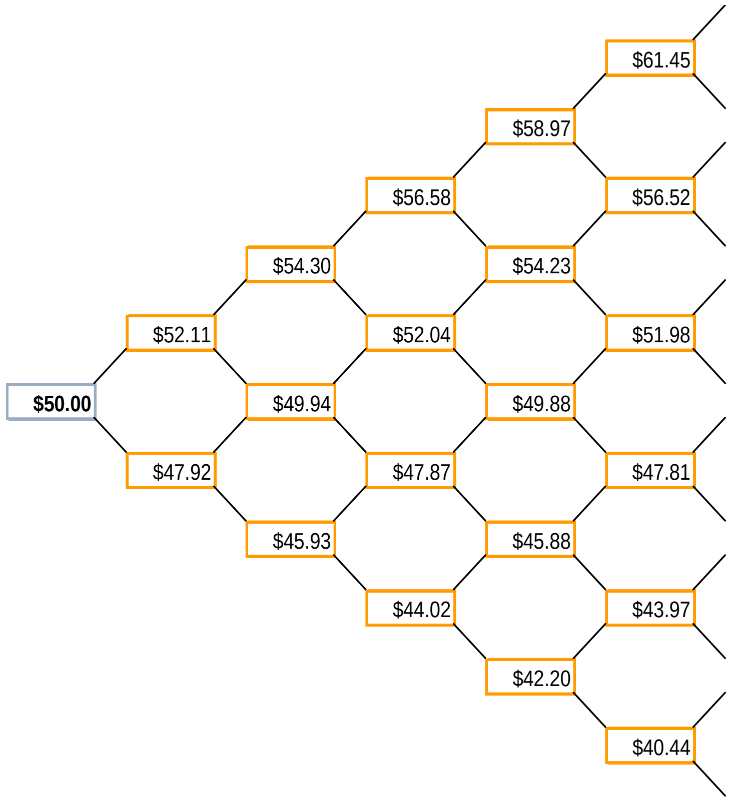
If market is valuing company's stock at 8 X EBITDA

More than $320 million in stock price value

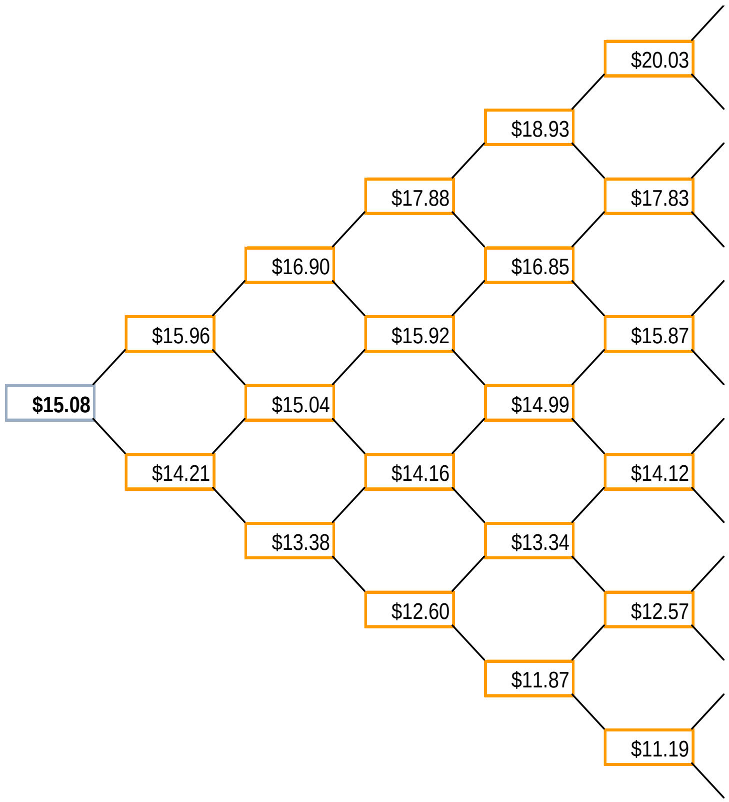
35% Volatility Stock Price Tree



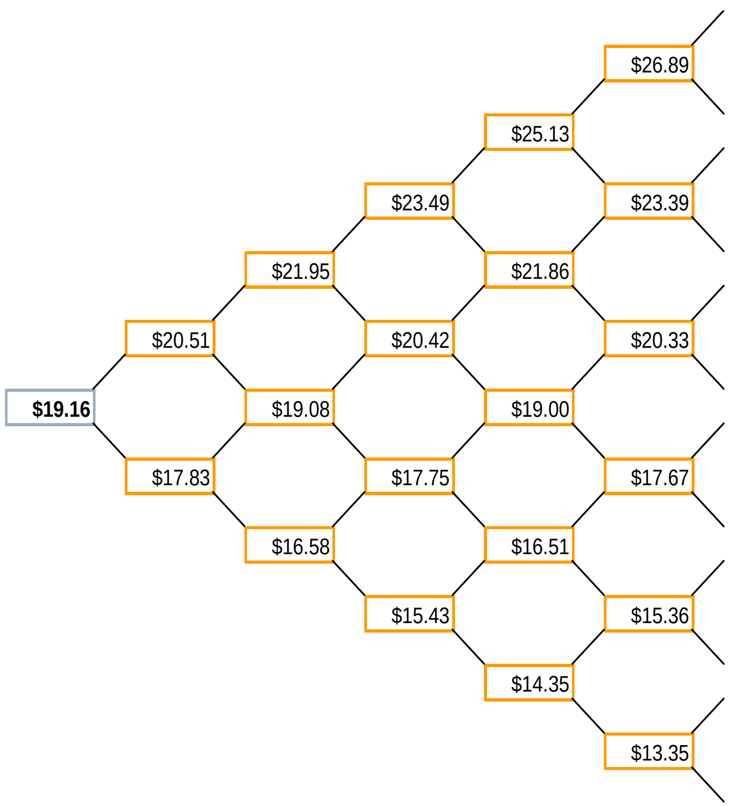
50% Volatility Stock Price Tree



35% Volatility Option Value Tree



50% Volatility Option Value Tree



Using History to Forecast the Future

History is informative to the extent that the spread of observed stock returns up to the present is predictive of the future spread of stock returns. It is important to consider qualifiers before using historical volatility to forecast future volatility, such as the following:

- If a company has not changed much over time, then a sample with more data is better.
- If a company has changed with respect to capital structure, lines of business, industry effects, etc., then observations from the distant past are less important. Additionally, implied volatility and comparable company data might be informative.
- If a company's recent history resulted in volatility that is unusually high (or low), then implied volatility and longer historical sampling might be necessary to make better forecasts that capture mean reversion.
- If there was unusual volatility related to a non-recurring event that is very likely not to repeat, then data may be omitted.
- If a company has only recently become public, then its history is inadequate for estimating volatility. Similar companies that are growing from newly public to more mature companies with more stock price history can be informative.
- For nonpublic companies FAS 123(R) recommends looking to estimates of volatility based on historical volatility of an appropriate set of comparable public companies. FASB refers to associated option valuation as the "calculated value method."

Using History to Forecast the Future

- How far in the past should we sample data? 1 year? 3 years? 4 years? 5 years?
- How frequently should we sample data? Daily? Weekly? Monthly?
- Should we put equal weight on all past returns? More weight on recent returns? More weight on returns that are not recent?

A common – and overused - rule of thumb: Match the historical sample period to the expected term of the option.

- In the case of Black-Scholes, this is a simple approach, but often a poor approach.
- In the case of a lattice or simulation model where expected term is a model output rather than a model assumption, this is a meaningless approach.

Constant vs. Time-varying Volatility

Fischer Black wrote:

Suppose we use the standard deviation . . .
of possible future returns on a stock . . . as a
measure of its volatility. Is it reasonable to
take that volatility as constant over time?
I think not.

Constant vs. Time-varying Volatility

- The Black-Scholes-Merton formula assumes that option exercises occur at the end of an option's contractual term, and that expected volatility, expected dividends, and risk-free interest rates are constant over the option's term.

- In contrast, a lattice model can be designed to accommodate dynamic assumptions of expected volatility and simulation is even more flexible.

Constant vs. Time-varying Volatility

- Black-Scholes requires a constant volatility input, which is estimated as an *average* of the spread of the returns that are observed over time, or an average from an implied volatility term structure.
- But the spread changes, often in unpredictable ways.
- The price of a stock might go through periods of low volatility and then suddenly experience volatility bursts that dissipate quickly, or that persist for a period of time.
- Changes in volatility correspond to company specific factors, or uncertainty in financial markets and the economy, generally.
- “Mean reversion” is the most commonly discussed pattern. This is the tendency for changes in volatility to dissipate in the long run so that that volatility tends to revert to a long run average, though it does not usually do so immediately after a burst of volatility. If volatility is estimated over a year or longer, large changes are found to be less common.
- But it important not to confuse structural change for a cycle within a mean reverting pattern. Think of Long Term Capital Management (“LTCM”) which made a highly leveraged bet on mean reversion.

Constant vs. Time Varying Volatility

- Use of very recent historical vols. exclusively should be supportable by fundamentals. Otherwise Company might regret the decision in the long run.
- Similarly, use of spot implied volatilities also might be good in the short-run but Company might have to live with the decision.
- Use of Spots and Lagged Volatilities likely smooths the estimate over time for companies without fundamental changes in capital or corporate structure.
- Auditors will have to be convinced that today's support for using spots does not conflict with tomorrow's support for using weighted averages.

Estimating Mean Reversion from Historical Data (GARCH)

- GARCH is an acronym for *generalized autoregressive conditional heteroskedasticity*. Heteroskedasticity refers to volatility that changes over time. The GARCH (*m*, *n*) model assumes there is a long run average volatility, or mean, to which volatility reverts. The current local volatility is the square root of a weighted average of the long run average squared volatility, recent squared returns, and recent squared volatility. The "*m*, *n*" refers to the number of return and volatility lags, respectively, that are used to explain the current local volatility.

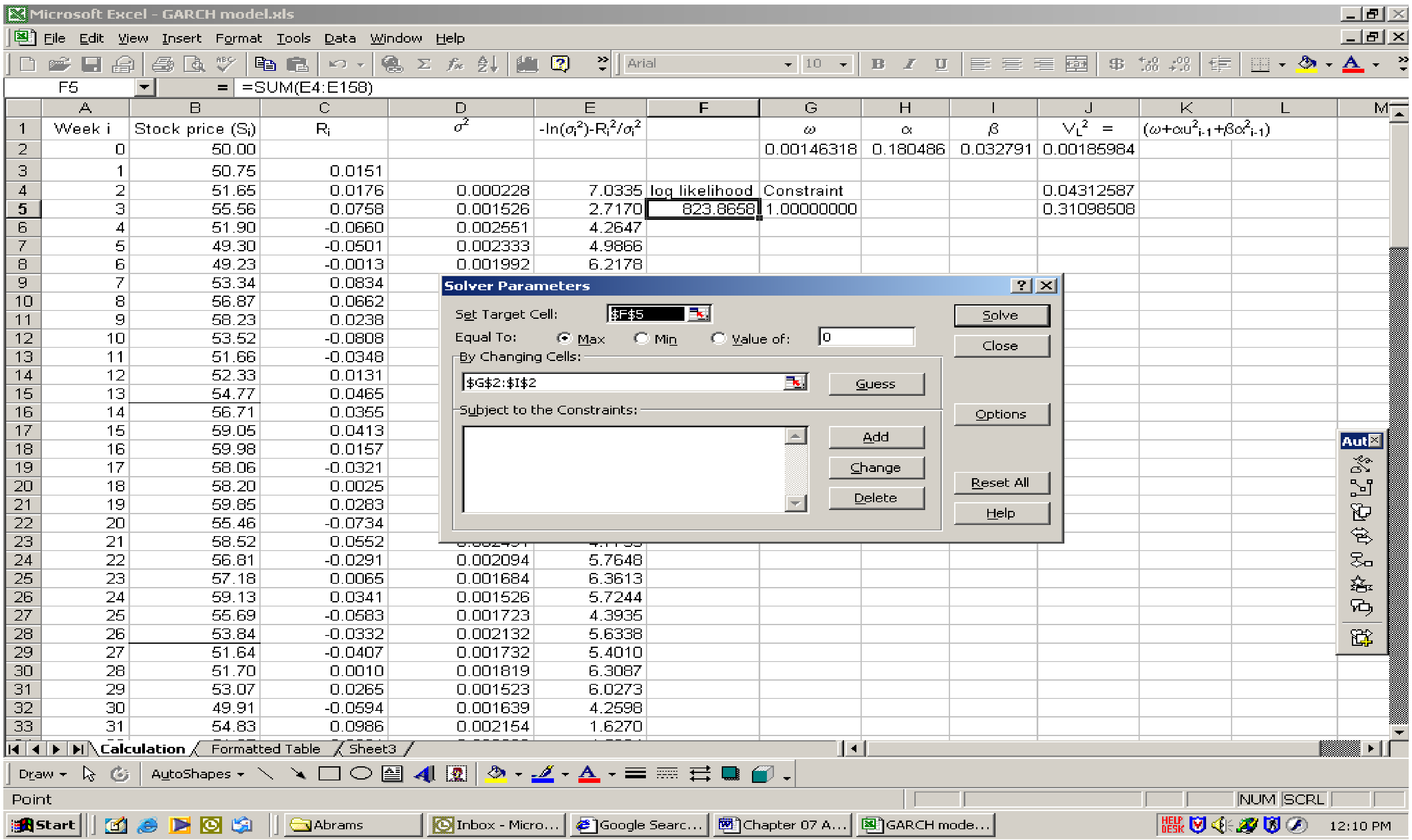
- The weights of the weighted average are estimated from historical data. Specifically, the GARCH (1, 1) model assumes that local volatility is defined by:

$$\sigma_n = \sqrt{(1-\alpha-\beta)V^2 + \alpha R_{n-1}^2 + \beta \sigma_{n-1}^2}$$

- To estimate the parameters, the first term is a constant, and then use numerical *maximum likelihood* methods to solve for the parameter values that best match the observed historical data. The resulting long run volatility, *V*, is implicitly estimated.

  - ω represents the weight given to the long run volatility.
  - α represents the extent to which volatility reacts to market movements.
  - β represents the extent to which volatility deviations from the long run average persist. It can be thought of as a decay rate.

Solving for GARCH Parameters



	A	B	C	D	E	F	G	H	I	J	K
1	Week i	Stock price (S_i)	R_i	σ^2	$-\ln(\sigma_i^2)-R_i^2/\sigma_i^2$		ω	α	β	V_L^2 =	$(\omega+\alpha u^2_{i-1}+\beta\sigma^2_{i-1})$
2	0	50.00					0.00146318	0.180486	0.032791	0.00185984	
3	1	50.75	0.0151								
4	2	51.65	0.0176	0.000228	7.0335	log likelihood	Constraint			0.04312587	
5	3	55.56	0.0758	0.001526	2.7170	823.8658	1.00000000			0.31098508	
6	4	51.90	-0.0660	0.002551	4.2647						
7	5	49.30	-0.0501	0.002333	4.9866						
8	6	49.23	-0.0013	0.001992	6.2178						
9	7	53.34	0.0834								
10	8	56.87	0.0662								
11	9	58.23	0.0238								
12	10	53.52	-0.0808								
13	11	51.66	-0.0348								
14	12	52.33	0.0131								
15	13	54.77	0.0465								
16	14	56.71	0.0355								
17	15	59.05	0.0413								
18	16	59.98	0.0157								
19	17	58.06	-0.0321								
20	18	58.20	0.0025								
21	19	59.85	0.0283								
22	20	55.46	-0.0734								
23	21	58.52	0.0552	[illegible]	[illegible]						
24	22	56.81	-0.0291	0.002094	5.7648						
25	23	57.18	0.0065	0.001684	6.3613						
26	24	59.13	0.0341	0.001526	5.7244						
27	25	55.69	-0.0583	0.001723	4.3935						
28	26	53.84	-0.0332	0.002132	5.6338						
29	27	51.64	-0.0407	0.001732	5.4010						
30	28	51.70	0.0010	0.001819	6.3087						
31	29	53.07	0.0265	0.001523	6.0273						
32	30	49.91	-0.0594	0.001639	4.2598						
33	31	54.83	0.0986	0.002154	1.6270						

Solving for GARCH Parameters

Estimation of parameters in GARCH model

Week *n*	**Stock price (S_n)**	***Return (R_n)***	σ_n^2	$-\ln(\sigma_n^2)-R_n^2/\sigma_n^2$
1	50.75			
2	51.65	0.017557727		
3	55.56	0.075834815	0.000308274	-10.57071037
4	51.90	-0.065981993	0.001185341	3.064838048
5	49.30	-0.050056259	0.001693966	4.901533258
6	49.23	-0.00131777	0.001829091	6.302986574
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
154	77.96	0.0305	0.000979	5.9814
155	78.48	0.0066	0.000996	6.8674
156	77.58	-0.0114	0.000874	6.8935

Estimated GARCH parameters				***Long Run***
ω	α	β	V_L^2 *(per week)*	***Avg. Volatility***
0.00146318	0.180485703	0.032790543	0.001859841	0.310985083

Estimating Mean Reversion from Historical Data

GARCH is a deterministic volatility; there are many sophisticated alternatives:

1. instantaneous volatility is a specified function of stock price and time
2. instantaneous volatility is an autonomous diffusion process
3. bounding quadratic variation
4. deterministic local volatility surface
5. stochastic forward local volatility surface
6. stochastic implied volatility surface

Some models can only be implemented with simulation while others can be implemented in lattices such as implied trees.

Dividends

- In estimating historical volatility, it is also necessary to adjust for discrete dividend payments during the sampling period.

- Price data is often dividend adjusted; if not, then dividend payments have to manually be added back to the stock price in the period immediately following the ex-dividend date. This adjustment is pre-tax, but it gives a reasonable approximation.

- Another method is to discard data for intervals that include ex-dividend dates.

Implied Volatility

- Implied volatility from observed prices of exchange-traded call and put options on a company's stock is the most important alternative to historical data.

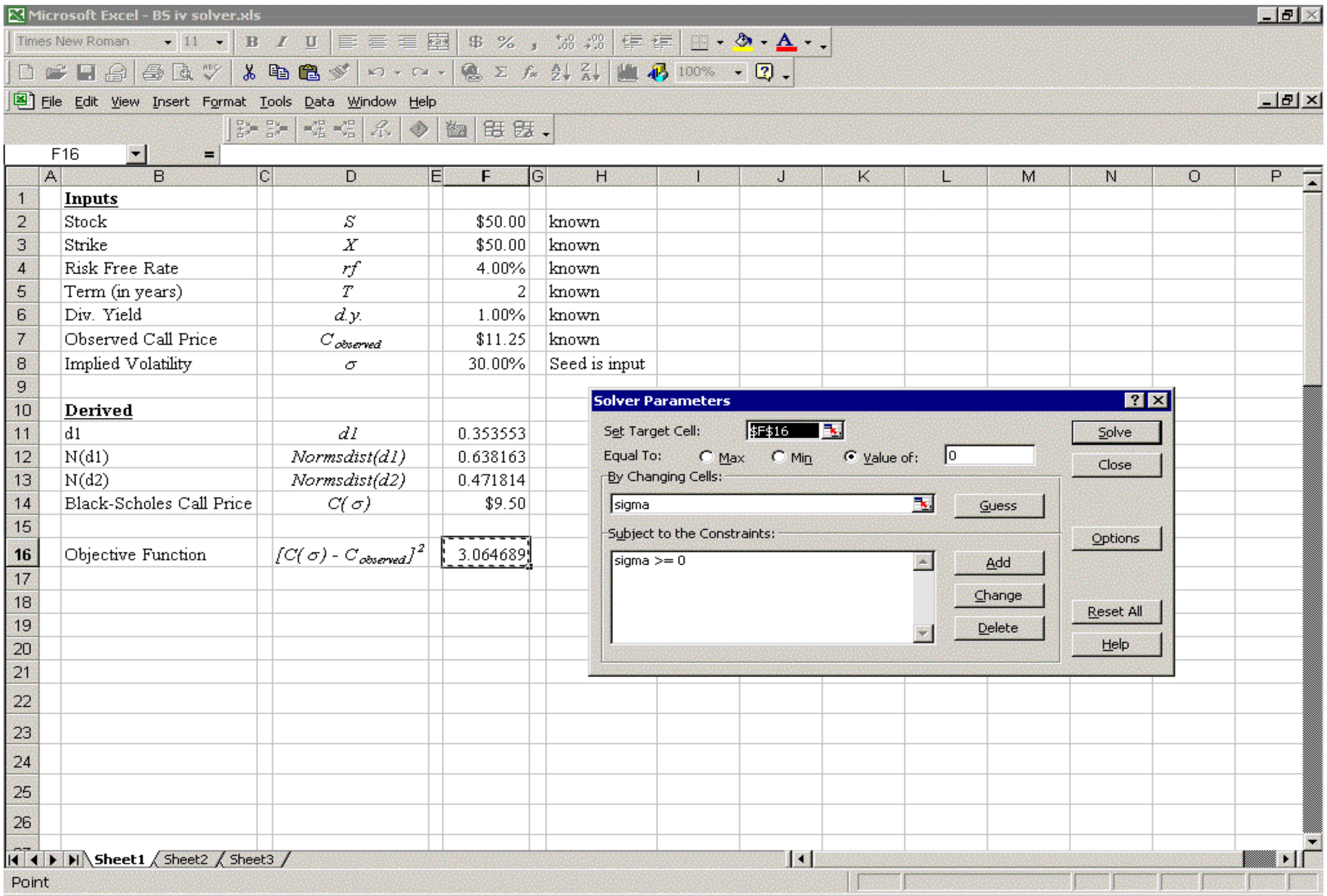
- If certain conditions are met, then implied volatility ought to impound the relevant information from historical volatility, as well as the market's assessment of current and future circumstances.

- In other words, implied volatility is forward-looking and ought to include at least as much information as historical volatility.

- The most relevant observed prices are those of short-lived options and long-term equity anticipation securities ("LEAP", a registered trademark of the Chicago Board Options Exchange). However, LEAPs generally have terms no more than 2 years.

- In theory, the prices of long-lived instruments such as warrants and convertible bonds are useful, but might not be practical due to lack of liquidity and other market imperfections.

Solving for Implied Volatility



Implied Volatility

There are three necessary conditions for implied volatility to provide a reliable forecasting tool:

1. There must be a liquid market for traded options, insuring that observed near-the-money prices are current and frequent. Stale prices and large bid-ask spreads can result in significant pricing errors.

2. The model that is used to infer volatility from observed prices must be well specified. This means, for example, that prices on American put options in a market with significant frictions should not be used to derive implied volatility with the Black-Scholes model.

3. The term and strike of the exchange-traded options should approximately correspond to the forecast.

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Implied Volatility

These 3 conditions are non-trivial:

- Intraday options market activity is not documented as well as stock market activity.
- The majority of exchange-traded options are short-term relative to employee stock options and thus, implied volatilities typically reflect short-term expectations.
- Furthermore, for a given term, implied volatility varies with strike price yielding a volatility “smile” or “skew.” In the case of a constant volatility forecast, where the volatility smile and volatility term structure are assumed away, there are several averaging considerations.

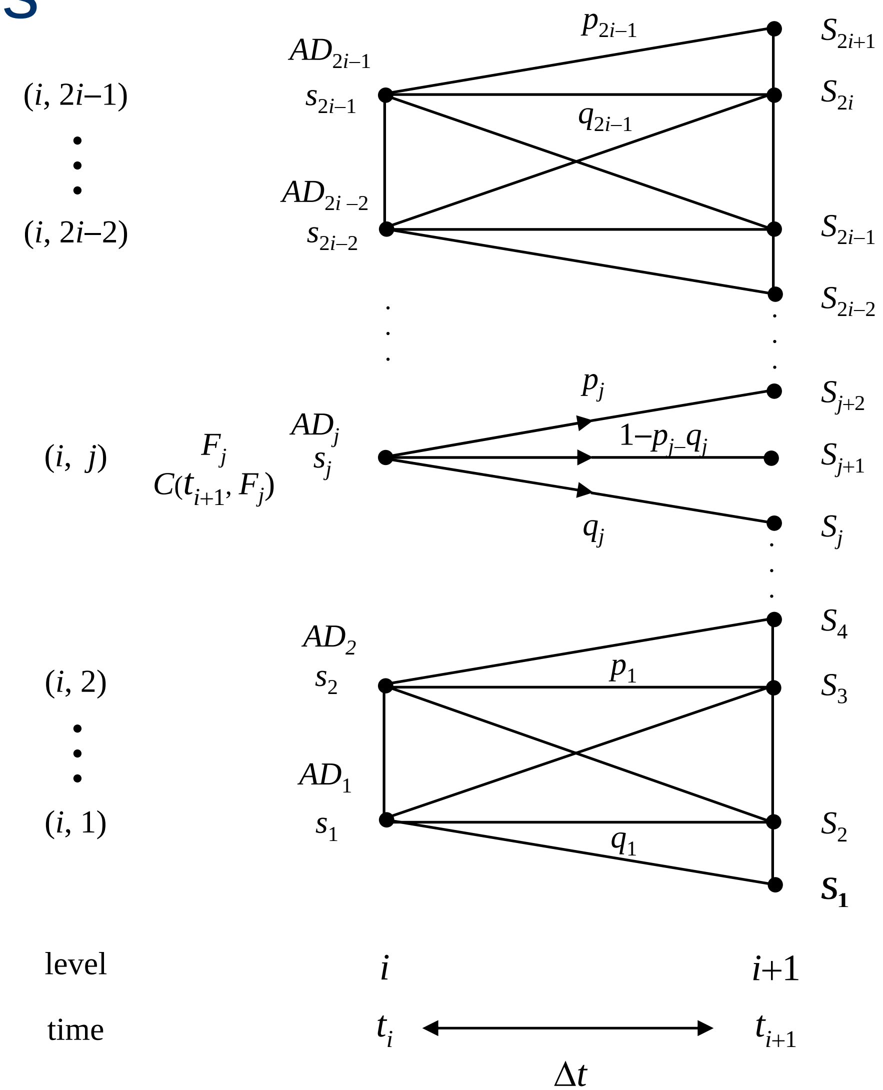
Using implied volatility properly requires expert guidance.

Implied Volatility Term Structure

Assume for this example that the dividend yield is at a constant continuous rate equal to 0.896% except for December and January options which will be assumed to have a zero dividend yield (because company XYZ does not go ex-dividend over this time). The expiration is the Saturday after the 3rd Friday of the contract month. The close of XYZ was 58.625 on November 22, 2004:

Month	Strike	Riskless Rate	Premium	TTM	Implied Vol
12/04	60	0.02908	6 ¼	0.079452	0.2279
01/05	60	0.03048	7	0.156164	0.3091
04/05	60	0.03163	8 ¼	0.405479	0.3087
01/06	60	0.033256	10 ¼	1.153425	0.2585
01/07	60	0.035056	13	2.150685	0.2334

Implied Trees



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